JGP - Vol. 5, n. 3, 1989

# A remark on the Gelfand-Tsetlin patterns for symplectic groups

A. A. KIRILLOV
Moscow State University
Moscow – USSR

*To my teacher I.M. Gelfand*
*for his 75th birthday*

**Abstract.** *In this paper, we study an approach by Gelfand-Tsetlin to the representation of symplectic groups.*

**1.** The theory of infinite dimensional representations has since its very beginning exerted influence on its classical prototype – the theory of finite dimensional representations. A clear example of this influence is the well known explicit construction by I. M. Gelfand and M. L. Tsetlin of a basis in the representation space for unitary and orthogonal groups. This construction is valid also for general linear groups $GL(n, \mathbb{C})$, for complex orthogonal groups $SO(n, \mathbb{C})$ and for all its real forms in view of H. Weyl's «unitary trick».

The elements of the basis in question are labelled by integer triangular matrices of a special kind – the so called Gelfand-Tsetlin patterns (see below). They are the discrete analogs of triangular subgroups involved in the realisation of infinite dimensional representations of the corresponding non-compact groups.

On the other hand the Gelfand-Tsetlin bases are analogous to the Young bases for irreducible representations of the symmetric groups. Just as the latter agree with the restrictions of the representations to the subgroups of the sequence

---

*Key-Words:* *symplectic groups, Gelfand-Tsetlin patterns.*
*1980 MSC:* *20 G 20, 20 G 40.*

(1) $$S_n \supset S_{n-1} \supset \ldots \supset S_2 \supset S_1,$$

the Gelfand-Tsetlin bases agree with the restrictions of representations to the subgroups of the sequences

(2) $$U(n) \supset U(n-1) \supset \ldots \supset U(2) \supset U(1)$$

and

(3) $$SO(n) \supset SO(n-1) \supset \ldots \supset SO(2) \supset SO(1).$$

The neighbouring subgroups in each of the sequences (1), (2), (3) possess the following remarkable property: every irreducible representation of the bigger group has simple spectrum viewed as a representation of the smaller group (dual property: every representation of the bigger group induced from an irreducible representation of the smaller group has simple spectrum). Such pairs of groups were investigated in representation theory for a long time, see e. g. Chapter 3 in the survey [2] and also [3], [4].

Unfortunately the sequence

(4) $$Sp(2n) \supset Sp(2n-2) \supset \ldots \supset Sp(4) \supset Sp(2)$$

does not possess this property. However, D. P. Zelobenko [5] managed to construct the bases in the representation spaces for symplectic groups which correspond to the hypothetical sequence of subgroups of the form

(5) $$Sp(2n) \supset ? \supset Sp(2n-2) \supset \ldots \supset ? \supset Sp(2).$$

I conjectured in [6] that the intermediate subgroups in (5) must be equal to the non-semisimple groups $TSp(2n)$ which are stabilizers of a vector in the standard realization of $Sp(2n+2)$. The group $TSp(2n)$ (triangular symplectic group) is the semi-direct product of $Sp(2n)$ by the $(2n+1)$-dimensional Heisenberg group $H_n$. In the recent paper [7] by I. M. Gelfand and A. V. Zelevinsky very apt notation $Sp(2n+1)$ was proposed for this group.

My conjecture was partly confirmed by V. V. Shtepin [8] but his result is rather cumbersome due to the lack of complete reducibility for finite dimensional representations of $Sp(2n+1)$. Moreover, this group has no compact form.

**2.** In this paper I want to draw attention to the other approach to Gelfand-Tsetlin patterns for symplectic groups. First, let us recall the classical results of Gelfand and

Tsetlin for the unitary group $U(n)$. Irreducible representations of $U(n)$ are labelled by highest weights, i. e. integer valued $n$-vectors $\mu = (m_1, m_2, \dots, m_n)$ satisfying the condition

$$m_1 \geq m_2 \geq \dots \geq m_n. \tag{6}$$

Under the restriction to $U(n-1)$ the irreducible representation $T_\mu^{(n)}$ of $U(n)$ decomposes without multiplicities and the spectrum of the restriction consists of those representations $T_\lambda^{(n-1)}$ for which the $(n-1)$-vector $(l_1, \dots, l_{n-1})$ satisfies the «betweenness condition»:

$$m_1 \geq l_1 \geq m_2 \geq l_2 \geq \dots \geq l_{n-1} \geq m_n. \tag{7}$$

This leads immediately to a labelling of a basis in the representation space for $T_\mu^{(n)}$ by the patterns of the form

$$\begin{array}{ccccccccccc}
m_1 & & m_2 & & m_3 & \dots & m_{n-1} & & m_n \\
& l_1 & & l_2 & & \dots & & l_{n-1} & \\
& & k_1 & & & \dots & k_{n-2} & & \\
& & & \ddots & & & & & \\
& & & & \ddots & & & &
\end{array} \tag{8}$$

(the so called Gelfand-Tsetlin patterns), where each member is an integer and is contained between the two above.

Irreducible representation of $Sp(2n)$ are also labelled by the highest weights of the form (6) with the supplement condition that $m_n \geq 0$. As Zhelobenko proved, the spectrum of the restriction of the irreducible representation $T_\mu^{(2n)}$ with highest weight $\mu$ of $Sp(2n)$ to the subgroup $Sp(2n-2)$ consists of representations $T_\lambda^{(2n-2)}$ with highest weights $\lambda = (l_1, l_2, \dots, l_{n-1})$ which can be included in the diagram

$$\begin{array}{ccccccccc}
m_1 & & m_2 & & m_3 & \dots & & m_n & \\
& p_1 & & p_2 & & \dots & p_{n-1} & & p_n \\
& & l_1 & & l_2 & \dots & & l_{n-1} &
\end{array}$$

satisfying the same betweenness condition and also the inequalities $p_n \geq 0$, $l_{n-1} \geq 0$.

This leads to a labelling of a basis in the representation space for $T_\mu^{(2n)}$ by patterns of the form:

$$\begin{array}{cccccccccccc}
m_1 & & m_2 & & m_3 & \dots & & m_{n-1} & & m_n & \\
& p_1 & & p_2 & & p_3 & \dots & & p_{n-1} & & p_n \\
& & l_1 & & l_2 & \dots & & l_{n-2} & & l_{n-1} & \\
& & & q_1 & & q_2 & \dots & & q_{n-2} & & q_{n-1} \\
& & & & k_1 & & k_2 & \dots & & k_{n-2} & \\
& & & & & \ddots & & \ddots & & \dots &
\end{array} \tag{9}$$

This pattern becomes more natural extended to the right by antisymmetry:

$$\begin{array}{ccccccccccccccccccccccc} m_1 && m_2 && m_3 \dots && m_n && 0 && -m_n && \dots -m_3 && -m_2 && -m_1 \\ & p_1 && p_2 && \dots p_{n-1} && p_n && -p_n && -p_{n-1} \dots && -p_2 && -p_1 & \\ && l_1 && l_2 \dots && l_{n-1} && 0 && -l_{n-1} && \dots -l_2 && -l_1 && \\ &&& q_1 && \dots q_{n-2} && q_{n-1} && -q_{n-1} && -q_{n-2} \dots && -q_1 &&& \\ &&&& k_1 \dots && k_{n-2} && 0 && -k_{n-2} && \dots -k_1 &&&& \end{array} \tag{10}$$

I shall call it «antisymmetric Gelfand-Tsetlin pattern». (This observation is made also in the recent paper [9]).

**3.** The choice of an orthogonal basis in the spaces of irreducible unitary representations of a compact group $G$ yealds also an orthogonal basis in the space $L_2(G)$. Namely to each pair of vectors $\zeta, \eta$ in the space of the representation $T$ there corresponds the «matrix element» $t_{\zeta,\eta}(g) = (T(g)\zeta, \eta)$ in $L_2(G)$.

In the case $G = U(n)$ a pair of vectors in the space of the representation $T^{(n)}$ is labelled by the pair of Gelfand-Tsetlin patterns with the same upper line. It is natural to glue these two patterns along the common line turning one of them upside down. If we rotate the pattern obtained $45°$ clockwise the labelling arise of an orthogonal basis in $L_2(U(n))$ by integer-valued $n$ by $n$ matrices satisfying the non-increasing conditions for lines and columns.

EXAMPLE 1. For $G = U(2)$ this basis consists of the functions

$$f_{\begin{pmatrix} m & p \\ q & n \end{pmatrix}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum \frac{a^\alpha b^\beta c^\gamma d^\delta}{\alpha!\beta!\gamma!\delta!} (ad - bc)^n, \tag{11}$$

where the sum is extended to all quadruples $(\alpha, \beta, \gamma, \delta)$ satisfying

$$\alpha + \beta = m - p, \quad \gamma + \delta = p - n$$

$$\alpha + \gamma = m - q, \quad \beta + \delta = q - n.$$

In particular, antisymmetric patterns correspond to the functions

$$f_{\begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{\alpha+\beta=m} \left(\frac{m!}{\alpha!\beta}\right)^2 \left(\frac{ad}{ad - bc}\right)^\alpha \left(\frac{bc}{ad - bc}\right)^\beta. \tag{12}$$

It is easy to check that the mapping $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{ad}{ad-bc} = \tau$ sends $U(2)$ into the segment [0, 1], and besides the Haar measure on $U(2)$ goes into the standard Lebesgue measure $d\tau$ on [0, 1]. Hence, polynomials

$$Q_m(\tau) = \sum_{\alpha+\beta=m} \left(\frac{m!}{\alpha!\beta!}\right)^2 \tau^\alpha(\tau - 1)^\beta$$

which are the images of $f\begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}$ differ only by a change of variable from the classical Legendre polynomials $P_m$ :

$$Q_m(\tau) = P_m(2\tau - 1).$$

The convolution on the compact group $G$ is expressed very simply in terms of matrix elements. Namely, the formula holds

$$t_{\zeta_1,\eta_1} * t_{\zeta_2,\eta_2} = \frac{(\zeta_2,\eta_2)}{\dim T} t_{\zeta_2,\eta_1}. \tag{13}$$

It implies the following recipe for computing the convolution of two functions of the type $f_M$. Let $M^+(M^-)$ denote the upper (lower) triangular part of the matrix $M$ the main diagonal included. Let $'$ denote transposing. Define $M_1 \circ M_2$ by the rule: if $M_1^+ = (M_2^-)'$ then $M_1 \circ M_2$ is obtained from $M_1^-$ and $M_2^+$ by glueing along the common diagonal; otherwise we set $M_1 \circ M_2 = \emptyset$ and $f_\emptyset = 0$. With this notation we have:

$$f_{M_1} * f_{M_2} = \text{const} \cdot f_{M_1 \circ M_2}. \tag{14}$$

It should be noted that the constant in (14) can be reduced to 1 by an appropriate choice of scaling of $f_M$. Thus, the basis $\{f_m\}$ gives an explicit isomorphism of the group algebra $C^*(U(n))$ with the direct product of matrix algebras $\prod_\mu \mathrm{Mat}_{d(\mu)}(\mathbb{C})$, where $d(\mu)$ is the dimension of the irreducible representation $T_\mu^{(n)}$ of $U(n)$, and $\mu$ runs through all the set of highest weights for $U(n)$.

**4.** Consider in the group algebra $C^*(U(n))$ the closed subspace $A_n$ spanned by the functions $f_M$ with antisymmetric (with respect to the second diagonal) matrix $M$. It follows from (14) that $A_n$ is a convolution subalgebra and from the discussion in $n^\circ 2$ it follows that $A_{2n+1}$ is isomorphic to $C^*(Sp(2n))$. The question naturally arises if one can give an inner characterisation of this subalgebra. We begin with a representation-theoretic interpretation of the antisymmetry condition.

THEOREM 1. *The following conditions are equivalent:*

(i) *The weight* $\mu = (m_1, \ldots, m_n)$ *is antisymmetric, i. e.* $m_k = -m_{n+1-k}, k = 1, 2, \ldots, n$.

(ii) *The irreducible representation* $T_\mu^{(n)}$ *of the group* $U(n)$ *with the highest weight* $\mu$ *is real, i.e. in the appropriate basis is written by real matrices.*

(iii) *The representation* $T_\mu^{(n)}$ *is equivalent to the contragredient representation* $\tilde{T}_\mu^{(n)}$.

(iv) *The representation* $T_\mu^{(n)}$ *is equivalent to the complex conjugate representation* $\bar{T}_\mu^{(n)}$.

(v) *The restriction of* $T_\mu^{(n)}$ *to the subgroup* $K_n = U(([\frac{n}{2}]) \times U([\frac{n+1}{2}])$ *naturally embedded in* $U(n)$ *has a fixed vector.*

(vi) *The representation* $T_\mu^{(n)}$ *occurs in the spectrum of the representation of* $U(n)$ *in the space* $L_2(X_n), X_n = U(n)/K_n$.

(vii) *All homogeneous Laplace-Casimir operators of odd degree vanish in the representation space of* $T_\mu^{(n)}$.

*Proof.* $(i) \Leftrightarrow (iii)$. The contragradient representation to $T_\mu^{(n)}$ has the hightest weight $-v$ where $v$ is the lowest weight of $T_\mu^{(n)}$. But the lowest weight can be obtained from the highest weight by the action of the element $w_0$ of the Weyl group having maximal length. For the group $U(n)$ the permutation $w_0$ acts as $w_0(k) = n+1-k$. Hence the condition $-v = w_0(\mu)$ is equivalent to the antisymmetry of $\mu$.

$(iii) \Leftrightarrow (iv)$. $T_\mu^{(n)}$ being unitary, the contragradient and complex conjugate representation coincide.

$(ii) \Leftrightarrow (iv)$. It is clear that (ii) implies (iv). Now, for every unitary representation $T$ in the Hilbert space $V$ the representation $T \otimes \bar{T}$ is naturally realised in the space $\operatorname{End} V$ by the formul $T \otimes \bar{T} : X \mapsto T(g)XT(g)^*$. The real subspace of $\operatorname{End} V$ consisting of hermitian matrices is evidently invariant under $T \otimes \bar{T}$. Recall now that the Cartan product $T_1 \boxtimes T_2$ of two irreducible representations of $U(n)$ is defined as the irreducible component of $T_1 \otimes T_2$ with maximal highest weight. If $v_i$ is a highest weight vector for $T_i, i = 1,2$ then $v_1 \otimes v_2$ is the highest weight vector for $T_1 \boxtimes T_2$. In the case $T_2 = \bar{T}_1$ we see that the highest weight vector for $T \boxtimes \bar{T}$ is the Hermitian matrix $X = v \otimes v^*$. Hence $T \boxtimes \bar{T}$ is real. The highest weight of $T_\mu^{(n)} \boxtimes \bar{T}_\mu^{(n)}$ is equal to $\mu - w_0(\mu)$ and therefore antisymmetric. Conversely, each antisymmetric highest weight $\lambda = (l_1, \ldots, l_2)$ can be written as $\mu - w_0(\mu)$ (we can put $\mu = (l_1, \ldots, l_{[\frac{n}{2}]}, 0, \ldots, 0)$.

$(v) \Leftrightarrow (vi)$. The Frobenius reciprocity law.

$(vi) \Rightarrow (ii)$. A well known theorem by E. Cartan about spherical functions on symmetric spaces.

$(ii) \Rightarrow (vi)$. Follows from the explicit description of the representation of $U(n)$ on $L_2(U(n)/K_n)$. The following more general fact is in [10], Cor. 4.2 of Ch. 5: The spectrum of the symmetric space $Gr_n^k = U(n)/U(k) \times U(n-k), k \leq \frac{n}{2}$, consists of the representations $T_\mu^{(n)}$ with antisymmetric $\mu$ of the form $\mu = (m_1, \ldots, m_k, 0, \ldots, 0, -m_k, \ldots, -m_1)$.

$(i) \Leftrightarrow (vii)$. The Harish-Chandra formula for the infinitesimal character of the irreducible representation in terms of the highest weight looks as follows (see [6] or [11]):

$$T_\mu(\Delta_p) = P(\mu + \rho) - P(\rho), \tag{15}$$

where $\Delta_P$ is the element in the centre of the enveloping algebra $U(\mathbf{u}(n))$ corresponding to an $U(n)$-invariant polynomial $P$ on $\mathbf{u}(n)^* \sim \mathbf{u}(n)$ via the Gelfand isomorphism, and where $\rho$ in our case is the vector $\left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2}\right)$ – the half-sum of the positive roots of $\mathbf{u}(n)$; $\mu$ and $\rho$ are identified with corresponding diagonal matrices from $gl(n, \mathbb{C}) = \mathbf{u}(n)^{\mathbb{C}}$. By the theorem by Chevally restrictions of $U(n)$-invariant polynomials on $\mathbf{u}(n)$ onto the subset of diagonal matrices are all $S(n)$-invariant polynomials on this set. Homogeneous polynomials of odd degree span the ideal with generators $\sigma_1, \sigma_3, \dots, \sigma_{1+2k}$, $k = \left[\frac{n-1}{2}\right]$, where $\sigma_k$ is the elementary symmetric functions $\sigma_k(\mu_1, \dots, \mu_n) = \sum_{i_1 < i_2 \dots < i_k} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_k}$. It easy to check that the zero set for this ideal consists of antisymmetric vectors and vectors obtained from them by permutations of coordinates. This yield the required assertion taking into account the antisymmetry of $\rho$. ∎

In the following we shall write $\Delta_k^n$ for the elements of $U(\mathbf{u}(m))$ corresponding to $\sigma_k$ and also identify $U(\mathbf{u}(m))$ with the subalgebra of $U(\mathbf{u}(n))$ for $m < n$.

The last assertion of the theorem allows one to define the subspace $A_n \subset C^*(U(n))$ by a system of differential equations. Namely, let for $X \in U(\mathbf{u}(n))$ $L_X(\text{resp}.R_X)$ denote the corresponding right (resp. left) invariant differential operator on $U(n)$ and write $L_k^n(\text{resp}.R_k^n)$ for $L_{\Delta_k^n}(\text{resp}.R_{\Delta_k^n})$.

COROLLARY. *The subspace $A_n \subset C^*(U(n))$ consists of all solutions $f$ of the system*

$$L_{1+2k}^m f = 0,\ R_{1+2k}^m f = 0,\ m = 1, 2, \dots, n;\ 0 \le k \le \left[\frac{m-1}{2}\right].$$

Note, that the conditions $L_1^m f = 0$, $m = 1, 2, \dots, n(\text{resp}.R_{1+2k}^m f = 0$, $m = 1, 2, \dots, n)$ are equivalent to the left (resp. right) invariance of $f$ under the subgroup $T^n$ of diagonal matrices in $U(n)$.

EXAMPLE 2. The subspace $A_2$ consists of two-sided invariant functions on $U(2)$ with respect to the subgroup $T^2$ of diagonal matrices. The subspace $A_3$ consists of those two-sides invariant functions on $U(3)$ with respect to the subgroup $T^3$ of diagonal matrices which satisfy the differential equation of third order $Mf = 0$ where $M = L_3^3 = R_3^3$.

**5.** There are two operations in the function space on a group: the multiplication and the convolution. They are related and this relation can be formulated in the language of Hopf algebras. (See e.g. [6], $n^\circ$ 12.3). For a compact Lie group $G$ the structure of a Hopf algebra can be introduced both in the space $\mathcal{D}(G)$ of smooth functions and in the dual space $\mathcal{D}'(G)$ of distributions. In the first case the operations are the usual multiplication $m$ and the so-called co-convolution

$$\delta : \mathcal{D}(G) \to \mathcal{D}(G) \widehat{\otimes} \mathcal{D}(G) \sim \mathcal{D}(G \times G)$$

given by

$$\delta\varphi(g_1, g_2) = \varphi(g_1 g_2).$$

In the second case the operations are the convolution $c$ and the co-multiplication

$$\Delta : \mathcal{D}'(G) \to \mathcal{D}'(G) \widehat{\otimes} \mathcal{D}'(G) \sim \mathcal{D}'(G \times G)$$

defined by the formula

$$\langle \Delta F, \varphi(g_1, g_2) \rangle = \langle F, \varphi(g, g) \rangle.$$

The operators $m$ and $\Delta$ as well as $c$ and $\delta$ are conjugates. The relation we have in mind can be expressed in the following form (which is one of axioms of Hopf algebras): $\delta$(resp.$\Delta$) is an algebra homorphism with respect to operations $m$ and $m \otimes m$ (resp. $c$ and $c \otimes c$). The Tannaka-Krein duality theorem (loc. cit.) gives essentially the device to recontruct the group $G$ from the Hopf algebra structure on $\mathcal{D}(G)$ or $\mathcal{D}'(G)$. This structure is defined besides relations (14) by so-called Clebsch-Gordan coefficients $C^M_{M_1\ M_2}$ expressing the multiplication in terms of the basis $\{f_M\}$ :

$$f_{M_1} \cdot f_{M_1} = \sum_M C^M_{M_1\ M_2} f_M. \tag{17}$$

Of course (17) is equivalent to the explicit decomposition of the tensor product of two irreducible representations of $U(n)$ into irreducible components because of the identity

$$t^{\pi_1}_{\zeta_1, \eta_1} \cdot t^{\pi_2}_{\zeta_2, \eta_2} = t^{\pi_1 \otimes \pi_2}_{\zeta_1 \otimes \zeta_2, \eta_1 \otimes \eta_2}.$$

In fact the quantities $C^M_{M_1\ M_2}$ correspond to the Clebsch-Gordan coefficients for special representations of $U(n) \times U(n)$, but its computation is equivalent to the computation of «classical» Clebsch-Gordan coefficient for $U(n)$. The result (rather cumbersome) can be found in [12], Ch. 7, §9.

It should be noted that the subspace $A_n$ introduced in Section 4 for $n > 2$ is not an algebra under the usual multiplication. However, for odd $n$ we can introduce therein a multiplication using the identification $A_{2n+1} \sim C^*[Sp(2n)]$. Thus, on $A_n$ with odd $n$ the structure of a commutative Hopf algebra is defined. Moreover, the Hopf algebras

$A_{2n+1}$ and $A_{2n-1}$ are connected by a natural projection corresponding to the inclusion $Sp(2n-2) \subset Sp(2n)$. This projection looks out very simply in terms of the basis $f_M$ and it leads to the natural

CONJECTURE. *There exists an operation* $\pi$ *on the set of admissible indices* $M$ *such that*

*a)* $\pi$ *sends an admissible* $n\times n$*-matrix* $M$ *to an admissible* $(n-1)\times(n-1)$*-matrix* $\pi(M)$ *or to* $\emptyset$.

*b) The linear operator* $P$ *defined by* $Pf_M = f_{\pi(M)}$ *is a projection of* $A_n$ *onto* $A_{n-1}$.

*c) There exists a Hopf algebra structure on* $A_n$ *for all* $n$ *(for* $n$ *odd it was introduced above) such that* $P$ *is a homomorphism with respect to multiplication on* $A_n$.

The last assertion yields the following property of the Clebsch-Gordan coefficients:

$$\sum_{\pi(M)=N} c^{M}_{M_1\,M_2} = c^{N}_{\pi(M_1)\pi(M_2)} \tag{18}$$

and the verification of this equation would be the first step in the proof of the conjecture.

It seems that the investigation of the Hopf algebras $A_n$ and also their non-commutati ve deformations in the spirit of [13] has a very interesting perspective in connection with quantum groups in the sense of Drinfeld [14].

## REFERENCES


[2] I. M. GELFAND, M. L. TSETLIN: *Finite dimensional representations of the group of unimodular matrices*, Doklady AN SSSR **71** (1950), 825-828; *Finite dimensional representations of the group of orthogonal matrices*, Doklady AN SSSR **71**, 1017-1020, 1950. (In Russian). See also I. M. Gelfand, Collected papers, Vol. 2, Springer Verlag, Berlin, 1988.

[2] A. A. KIRILLOV: *Introduction to the theory of representations and non-commutative harmonic analysis*, Itogi Nauki i techn., VINITI, Sovr. probl. matem. Fund. napravleniya, **22**, 5-162, 1988. (English translation by Springer Verlag under the title *Encyclopedia of Mathematical Sciences*, Vol. 22, to appear).

[3] M. KRÄMER: *Multiplicity free subgroups of compact connected Lie groups.* Arch. Math., 1976, Bd. 27, No. 1, 28-36.

[4] I. V. MIKITYUK: *On integrability of invariant Hamiltonian system with homogeneous configurations spaces*, Matem. Sb. **129**, No. 4, 559-577, 1986. (In Russian).

[5] D. P. ZHELOBENKO: *Compact Lie groups and their representations.* Providence R. I., AMS, 1973.

[6] A. A. KIRILLOV: *Elements of the theory of representations.* Springer Verlag, Berlin, 1976.

[7] I. M. GELFAND, A. V. ZELEVINSKY: *Models of representations of classical groups and their hidden symmetries*, Funkt. anal. i ego pril. **18**, No 3, 14-31, 1984. (In Russian. See also I. M. Gelfand, *Collected papers*, Vol. II, Springer Verlag, Berlin, 1988).

[8] V. V. SHTEPIN: *Splitting of multiple points of the spectrum in the reduction* $Sp(2n) \downarrow Sp(2n-2)$, Funkt. anal. i ego pril. **20**, No. 4, 95-97, 1986; *On a class of finite dimensional* $Sp(2n-1)$ *-modules,* Uspekhi Mat. Nauk. **41**, No. 2, 287-288.
[9] R. A. PROCTOR: *Odd symplectic groups,* Invent. Math. **92**, 307-332, 1988.
[10] S. HELGASON: *Groups and geometric analysis.* Academic Press, 1984.
[11] J. DIXMIER: *Algèbres enveloppantes.* Gauthier-Villars, Paris, 1974.
[12] A. U. KLIMYK: *Matrix elements and Clebsch-Gordan coefficients for group representations.* Naukova Dumka, Kiev, 1974. (In Russian).
[13] S. L. WORONOWICZ: *Compact matrix pseudogroups,* Commun. Math. Phys. **111**, 613-665, 1987.
[14] V. G. DRINFELD: *Quantum groups,* Zap. Nauchn. Scmin. LOMI **155**, 18-49, 1986. (In Russian. See also ICM Proceedings, Berkeley, 1986).

*Manuscript received: October 18, 1988.*